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On K-Flows and Irreversibility[1]

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Here we give the proof of a general theorem concerning irreversibility that was stated earlier by Misra and Prigogine. In terms of a unitary group describing a deterministic dynamical evolution and a related Markov semigroup describing an associated coarse grained probabilistic evolution, it is shown that the original dynamics are necessarily those of a K flow. Thus a reversible dynamics which permits such an intertwining to an irreversible description must possess a high degree of instability.

KEY WORDS: Kolmogorov flows; coarse graining; dynamical system; intertwining.

1. INTRODUCTION

In the foundations of kinetic theory a traditional procedure for constructing a probabilistic Markovian master equation from a given deterministic dynamics involved two steps: (a) impose a "coarse graining," and (b) go to a weak coupling limit to obtain the Markov property. Among the objections to this procedure are a lack of precise rigor in (a) and the approximate nature of the limit in (b).

In Ref. 1 an approach is described whereby an exact Markovian master equation is obtained directly from a projection which intertwines the deterministic and probabilistic descriptions. The approach[1] depends on the following theorem. In the theorem, U_t is a unitary evolution representing in state space a given deterministic measure preserving reversible dynamics T_t, and W_t is to be a strongly irreversible Markov

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semigroup representing a probabilistic description brought about by a "coarse graining" projection P.

Theorem. For the existence of a "coarse graining" of T_t implemented by a projection P satisfying the conditions:

(i) $PU_t\rho = PU_t\rho'$ for all $t \Rightarrow \rho = \rho'$

(ii) $PU_t = W_t^* P$ for $t \geq 0$

(iii) P is a positivity preserving self-adjoint projection

(iv) $P(1) = 1$

it is necessary and sufficient that the dynamical system be a K flow.

In the Theorem, the strongly irreversible Markov semigroup W_t must satisfy the four conditions:[1]

(i′) $\|W_t^*\rho - 1\| \to 0$ monotonically as $t \to \infty$ $\forall \rho \geq 0$ with $\int \rho d\mu = 1$

(ii′) $W_t^*(1) = 1$

(iii′) W_t is a positivity preserving contraction semigroup

(iv′) $W_t(1) = 1$

The general relation of irreversibility to the existence of coarse grained or similarity changes of representation is an intriguing question with important physical and mathematical implications.[1–8] The sufficiency of the Theorem was demonstrated in Ref. 6. Our proof of the necessity depends on a connection to the theory of conditional expectation.[9]

2. COARSE-GRAINED INTERTWINING NECESSITATES *K*-FLOW DYNAMICS

We recall the setting.[1,6] One is given a deterministic dynamics T_t in a physical phase space Γ described by a unitary evolution U_t in a state space $\mathscr{L}^2(\Gamma, \mathscr{B}, \mu)$ according to

$$(U_t f)(x) = f(T_t x) \tag{1}$$

$x \in \Gamma$, $f \in \mathscr{L}^2(\Gamma, \mu)$. A coarse graining projection P is applied to the evolution, yielding the description

$$W_t^* = PU_t P \tag{2}$$

See Ref. 6 for the proof that for any K flow dynamics the coarse-grained evolution (2) possesses the properties stated in the Theorem.

To establish here the Converse, namely, that under the conditions of the Theorem, the dynamics is necessarily that of a K flow, we recall that a

K flow is a dynamical system $(\Gamma, \mathscr{B}, \mu, T_t)$ with an order structure connecting σ subalgebras $\mathscr{B}_t$ and a measure preserving dynamics T_t satisfying the properties:

(a) $T_t\mathscr{B}_0 = \mathscr{B}_t \subseteq \mathscr{B}_s = T_s\mathscr{B}_0,\ t \leqslant s$

(b) $\bigvee_{t=-\infty}^{t=\infty} \mathscr{B}_t = \mathscr{B}$

(c) $\bigcap_{t=-\infty}^{t=\infty} \mathscr{B}_t = \mathscr{B}_{-\infty}$

where $\mathscr{B}_{-\infty}$ is the trivial σ algebra generated by Γ and sets of measure 0. Here the space Γ is assumed compact and μ is a positive measure normalized to $\mu(\Gamma) = 1$. All $\mathscr{B}_t$ are assumed separable.

Proof of the necessity. We let

$$P_{-t} = U_{-t} P U_t \tag{3}$$

for all t. With this family $P_t = U_t P U_{-t}$ we wish to construct a K flow.

Consider first the case $P = P_0$. Clearly[9] P_0 is a conditional expectation and we thus know that there exists a (unique) σ algebra $\mathscr{B}_0$ such that the range of P is $\mathscr{L}^2(\Gamma, \mathscr{B}_0, \mu)$. Likewise, using $U_t(1) = 1$, and the positivity of U_t, all P_t are seen to be conditional expectations with ranges $\mathscr{L}^2(\Gamma, \mathscr{B}_t, \mu)$.

Let us next prove that the ranges $\mathscr{L}^2(\Gamma, \mathscr{B}_t, \mu)$ are the "correct" ranges for a K flow with underlying dynamics defined by $\mathscr{L}_t = T_t(\mathscr{B}_0)$, i.e., that the $\mathscr{B}_t$ induced by the P_t are consistent with those induced from the original dynamics. Let Q be the orthogonal projection of $\mathscr{L}^2(\Gamma, \mathscr{B}, \mu)$ onto $\mathscr{L}^2(\Gamma, T_t(\mathscr{B}_0), \mu)$. P_t and Q are orthogonal projections, so to show they are equal we need only show they have the same ranges. Take f in $R(Q)$, then f is measurable with respect to the σ algebra $T_t(\mathscr{B}_0)$. From the definition of U_t we see $(U_{-t}f)(s) = f[T_t(s)]$ is measurable, as the composition of two measurable functions, with respect to $\mathscr{B}_0$. Then $PU_{-t}(f) = U_{-t}(f)$, and $P_t(f) = U_t U_{-t}(f) = f$, or f is in $R(P_t)$. A similar argument shows $R(P_t) \subseteq R(Q)$.

Now that we have identified P_t the imprimitivity condition

$$U_t^* P_\lambda U_t = P_{\lambda - t} \tag{4}$$

is easily checked. We next show the conditions (a), (b), and (c) of a K flow are satisfied.

To prove the monotone property (a) we first prove a special case. For $s \geqslant 0$ using (ii) we have $P_{-s}P = U_{-s}PU_sP = U_{-s}W_s^*P = U_{-s}PU_s = P_{-s}$, or $P_{-s}P = P$ if $s \geqslant 0$. Suppose $s < t$. $U_t^*P_sP_tU_t = U_t^*P_sU_tU_t^*P_tU_t = P_{s-t}P = U_t^*P_sU_t$ which follows from (4) above. Thus $P_sP_t = P_s$ for $s \leqslant t$. This implies $\mathscr{B}_s \subseteq \mathscr{B}_t$ in the sense that every element of $\mathscr{B}_s$ differs from an element of $\mathscr{B}_t$ by at most a set of measure zero. This gives (a) in the definition of a K flow.

To establish (b) we let $\mathscr{A} = \bigvee_{t=-\infty}^{\infty} \mathscr{B}_t$. Then $\mathscr{A} \subseteq \mathscr{B}$ in the above sense, but suppose there exists a set in $\mathscr{B}$ that is not equal almost everywhere to a set in $\mathscr{A}$. Then $\mathscr{L}^2(\Gamma, \mathscr{A}, \mu)$ is properly contained in $\mathscr{L}^2(\Gamma, \mathscr{B}, \mu)$. Pick $f \in \mathscr{L}^2(\Gamma, \mathscr{B}, \mu)$ such that f is orthogonal to $\mathscr{L}^2(\Gamma, \mathscr{A}, \mu)$. Then $(f, P_t(g)) = 0$ for all t and all g, or $P_t(f) = 0$ for all t. But $U_t^* P U_t(f) = P_{-t}(f) = 0$, and $PU_t(f) = U_t P_{-t}(f) = 0$. By (i) this implies $f = 0$, and $\mathscr{B} = \bigvee_{t=-\infty}^{\infty} \mathscr{B}_t$.

To verify (c) let $\mathscr{B}_{-\infty} = \bigcap_{t=-\infty}^{\infty} \mathscr{B}_t$. Take $f \in \mathscr{L}^2(\Gamma, \mathscr{B}_{-\infty}, \mu)$ with $f \geqslant 0$, $f \neq 0$. Let $g = f/(\int f d\mu)$ as before so that $\int g d\mu = 1$. Condition (i′) implies $\| W_t^* g - 1 \| \to 0$ as $t \to \infty$. But $\mathscr{B}_{-\infty} \subseteq \mathscr{B}_t$ for all t, and so for all t we have $P_t(g) = g$ and $W_t^*(g) = W_t^* P(g) = PU_t(g) = U_t P_{-t}(g) = U_t(g)$. In order for $U_t(g)$ to converge to 1 in $\mathscr{L}^2$ norm it must be true that $g = 1$ a.e., because $\| U_t(g) - 1 \| = \| g - 1 \|$. We have shown the only $f \geqslant 0$ with $f \neq 0$ in $\mathscr{L}^2(\Gamma, \mathscr{B}_{-\infty}, \mu)$ are constants. This implies $\mathscr{B}_{-\infty}$ is the trivial σ algebra generated by sets of measure zero.

Remark. It is not yet clear which dynamics are intrinsically necessary in intertwined similarity changes of representation $W_t = \Lambda U_t \Lambda^{-1}$.[1-8]

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